where C^* is a $1\times n$ matrix found so that the eigenvalues of the closed-loop system matrix of

$$\dot{x} = Ax + B^*u \tag{11}$$

$$u = C^*x \tag{12}$$

where

$$B^* = (B_1 + \theta_2 B_2 + \theta_3 B_3 + \cdots + \theta_m B_m) \tag{13}$$

take on the preassigned eigenvalues. This is always possible to do since (11) is controllable [2] (the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_\alpha$ are pairwise distinct and the vector $\xi = T^{-1}B^*$ has every component corresponding to the last row of each Jordan block of J nonzero) and there is a single input only to the system [3].

Then the eigenvalues of the closed-loop system matrix obtained by using (9) with (1) will take on the preassigned values since

$$(A+BC)$$

$$= \left\{ A + (B_1, B_2, \cdots, B_m) \begin{bmatrix} C^* \\ \theta_2 C^* \\ \theta_3 C^* \\ \vdots \\ \theta_m C^* \end{bmatrix} \right\} \tag{14}$$

$$= (A + B^*C^*). \tag{15}$$

Consider now the case when $\lambda_1, \lambda_2, \cdots, \lambda_\alpha$ are not pairwise distinct. It will be shown that a linear feedback control system can always be found so that (5) is transformed into a system with n distinct eigenvalues. This means (using the results just obtained), that a linear feedback control system can then be applied so that the eigenvalues of the closed loop system matrix take on n preassigned values, thus proving the necessity of the theorem.

Transform the system (5) by:

$$y = T^* \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{16}$$

so that

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \begin{matrix} z_1 \\ z_2 \end{matrix} + \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} u \tag{17}$$

where J_1 contains all the distinct eigenvalues of A and J_2 contains all the remaining eigenvalues of A.

Now J_2 contains $s \leq \alpha$ Jordan blocks of J, some of which are repeated. Let the last Jordan block of J_2 be of dimension r_k. Apply now a linear feedback control system to (17):

$$u = [0, (0, k)] \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{18}$$

where k is an $m\times r$ matrix, so that the r_k eigenvalues of the last Jordan block of J_2 are put equal to r_k distinct values not equal to the eigenvalues of J_1. This is always possible to do since the subsystem of (17) corresponding to the last Jordan block of J_2 is controllable and has a minimal polynomial of degree r_k.

Consider now the system obtained on applying such a control system:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} J_1, & {}_1(0,k) \\ 0, & J_2 + {}_2(0,k) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} u. \tag{19}$$

This has r_k more distinct eigenvalues than (17). A new linear feedback control system can now be applied to (19) by transforming the system (19) into Jordan canonical form and then repeating the process of (16)–(19) so that the eigenvalues of the last Jordan block of the new J_2 matrix obtained take on distinct eigenvalues not equal to the distinct eigenvalues of the system matrix of (19). This process is then repeated until the system matrix finally obtained has n distinct eigenvalues.

The necessity of the theorem has, therefore, been proved.

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Remarks by W. M. Wonham[1]

Davison's alternative proof of the fact that controllability implies pole assignability is welcome. Whether or not his proof (using the Jordan form of A) is really simpler is perhaps a matter of taste, since the Jordan form is much more sophisticated than the block triangular form used by Langenhop and by me, and exhibits much more of the structure of A than one really needs to solve the problem. Of course, all these methods are useful in applications.

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[1] Manuscript received July 11, 1968.

Comments "On Pole Assignment in Multi-Input Controllable Linear Systems"

Abstract—A short and direct new proof is given to Wonham's theorem that a time invariant multi-input linear dynamical system is controllable only if its poles can arbitrarily be reassigned in a closed-loop system by means of a constant (state variable) feedback law. The construction provided in the proof is directly applicable as an effective algorithm for this pole assignment.

In the above paper,[1] Wonham investigated the following problem. Let

$$\dot{x} = Ax + Bu \tag{1}$$

be a constant parameter linear dynamical system with x an n-dimensional state vector, u an m-dimensional input vector, and A and B $n\times n$ and $n\times m$ constant real matrices, respectively. The closed-loop system with $u = Cx + v$ is given as

$$\dot{x} = (A + BC)x + Bv \tag{2}$$

where C is a constant $m\times n$ feedback matrix and v is a new input vector.

Let L denote the collection of all sets of n complex numbers $\Lambda = \{\lambda_1, \cdots, \lambda_n\}$ such that if $\lambda_i \epsilon \Lambda$ and $\mathrm{Im}(\lambda_i) \neq 0$, then $\bar{\lambda}_i \epsilon \Lambda$, where $\bar{\lambda}_i$ is the complex conjugate of λ_i.

The theorem proved by Wonham is the following.

Theorem: The system (1) is controllable 1) if and 2) only if for every $\Lambda \epsilon L$ there exists a real matrix C such that (2) has Λ as its set of poles (i.e., Λ is the set of eigenvalues of $A+BC$).

The proof of 1) given by Wonham is straightforward, but the proof of 2), on which the actual construction of C hinges, is complicated, depends on a certain canonical form of Langenhop[2] (see also Luenberger[3]), and is not entirely constructive.

The purpose of this correspondence is to exhibit a short direct proof of 2) which is considerably simpler than Wonham's, depends on no canonical form, and is constructive.

The central construction of the present proof of 2) is contained in the following lemma.

Lemma: Let (A, B) be a controllable pair and let $b_1, \cdots, b_m$ be the column vectors of B. Then for any $i = 1, \cdots, m$ $(b_i \neq 0)$ there exists a matrix C_i such that $(A + BC_i, b_i)$ is controllable.

Proof: Recall that (A, B) is controllable if and only if the $n\times nm$ matrix

$$K = [B, AB, \cdots, A^{n-1}B]$$

has rank n. Then, since (A, B) is controllable (see, e.g., Luenberger[3]), there exist distinct columns $b_{i_1}, \cdots, b_{i_p}$ of B $(p \leq m)$ with $i_1 = i$, and integers

$$k_{i_1}, \cdots, k_{i_p} \geq 1, \; \sum_{j=1}^{p} k_{i_j} = n,$$

such that the set of n vectors

$$V_i = \{b_{i_1}, Ab_{i_1}, \cdots, A^{k_{i_1}-1}b_{i_1}, b_{i_2}, Ab_{i_2}, \cdots, A^{k_{i_2}-1}b_{i_2}, \cdots, A^{k_{i_p}-1}b_{i_p}\}$$

forms a basis for R^n, and for each $j = i_1, \cdots, i_p$ the vector $A^{k_j}b_j$ is a linear combination of the preceding basis elements. From here on, for simplicity of notation, the subscript i will be dropped, and it will be assumed, without loss of generality, that $i_1 = 1$, $i_2 = 2, \cdots, i_p = p$ (this can always be accomplished by interchanging the columns of B which is equivalent to interchanging the indices of the components of u). Let Q be the $n\times n$ (nonsingular) matrix of vectors of V,

$$Q = [b_1, Ab_1, \cdots, A^{k_1-1}b_1, \cdots, A^{k_p-1}b_p],$$

and define the $m\times n$ matrix C as

$$C = SQ^{-1}, \tag{3}$$

where S is an $m\times n$ matrix with columns s_l, $l = 1, \cdots, n$, defined as

$$s_{l\cdot} = e_{j+1} \qquad \text{for } j = 1, \cdots, p-1,$$

where

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$$t_j = \sum_{r=1}^{j} k_r$$

and e_{j+1} is the $(j+1)$th column of the $m \times m$ unit matrix, and $s_l = 0$ otherwise.

It will now be shown that C as defined above actually satisfies the conditions of the lemma. Since $CQ = S$, it is readily seen that

$$CA^{k_j-1}b_j = e_{j+1} \qquad \text{for } j = 1, \cdots, p-1$$

and

$$CA^r b_j = 0$$

where $0 \leq r \leq k_j - 2$ for $j = 1, \cdots, p-1$, and $0 \leq r \leq k_p - 1$ for $j = p$.

Let $\hat{A} = A + BC$ and let $\hat{Q}$ be the "controllability matrix" of $(A+BC, b_1)$ given as

$$\hat{Q} = [b_1, \hat{A}b_1, \cdots, \hat{A}^{n-1}b_1].$$

To see that $\hat{Q}$ has rank n, note that the vectors $\hat{A}^i b_1$, $j = 0, \cdots, n-1$ are given as

$$\begin{aligned}
b_1 &= b_1 \\
\hat{A}b_1 &= (A + BC)b_1 = Ab_1 \\
\hat{A}^2 b_1 &= (A + BC)Ab_1 = A^2 b_1 \\
&\cdots\cdots\cdots\cdots \\
\hat{A}^{k_1-1}b_1 &= A^{k_1-1}b_1 \\
\hat{A}^{k_1}b_1 &= (A + BC)A^{k_1-1}b_1 \\
&= Be_2 + A^{k_1}b_1 = b_2 + \cdots \\
\hat{A}^{k_1+1}b_1 &= (A + BC)(b_2 + \cdots) \\
&= Ab_2 + \cdots \\
&\cdots\cdots\cdots\cdots \\
\hat{A}^{n-1}b_1 &= (A + BC)(A^{k_p-2}b_p + \cdots) \\
&= A^{k_p-1}b_p + \cdots
\end{aligned}$$

where, in the above expressions, $\cdots$ denotes linear combinations of the preceding vectors. Clearly, these vectors are linearly independent by the independence of the vectors of V. Q.E.D.

The proof of 2) is now immediate.

Proof of 2): Since (A, B) is controllable, construct, according to the above lemma, a matrix, say, C_1, such that $(A + BC_1, b_1)$ is controllable. Since the theorem is well known for systems with scalar control (see, e.g., Brockett[4]) one can readily find an n-vector b such that $A + BC_1 + b_1 b^T$ has Λ as its set of eigenvalues. The desired feedback matrix C is then given by $C_1 + \hat{C}$ where $\hat{C}$ is the $m \times n$ matrix which is zero except for the first row which is b^T. Q.E.D.

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Author's Reply[6]

Heymann's alternative proof of the fact that controllability implies pole assignability is nice and neat. A simple and useful generalization of Heymann's lemma is the following. Write $\mathcal{B}$ for the range of B, $\{b\}$ for the subspace along b, and

$$\{A \,|\, \mathcal{B}\} = \mathcal{B} + A\mathcal{B} + \cdots + A^{n-1}\mathcal{B}.$$

Then for every nonzero $b \in \mathcal{B}$, there exists C such that

$$\{A + BC \,|\, \{b\}\} = \{A \,|\, B\}.$$

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[6] Manuscript received July 11, 1968.

Adaptive Tracking of Maneuvering Targets

***Abstract*—A means is suggested heuristically by which Kalman sequential estimation can be made adaptive to target maneuvers without the sacrifice of tracking accuracy in the nonmaneuvering portions of a trajectory. The adaptation requires backsliding in the gain schedule and reprocessing of the most recent several measurements. These steps are initiated by a maneuver detector which senses a buildup of bias in the filter's estimates.**

INTRODUCTION

The discrete Kalman filter has found successful application in the processing of radar data for trajectory and orbit determination, and in other tracking problems where the object being tracked (the target) is not subjected to what might be termed unknown deterministic forcing, i.e., maneuvers.

This correspondence suggests heuristically a means by which Kalman sequential estimation can be made adaptive to target maneuvers without the sacrifice of tracking accuracy in the nonmaneuvering portions of the trajectory. The adaptation has two aspects, one of regression, or backsliding in the gain schedule, the other of reprocessing the most recent several measurements. These measures are initiated by a maneuver detector which senses a buildup of bias in the filter's estimates.

MANEUVERS

From the point of view of a target pilot, maneuvers result from the application of known deterministic forcing functions. From the point of view of a radar-equipped observer, maneuver motion may be observed, but the explicit forces causing the maneuver and the precise time of their application can never be known. To the observer, then, these forces are unknown, but nonetheless retain their deterministic character. Attempts have been made to embed the effects of maneuvers in the covariance matrix of the random excitation provided for in the filter structure. Such embedding has obvious drawbacks; it assumes that maneuvers are random in nature, and it seriously degrades estimation accuracy by overweighting raw data for nonmaneuvering trajectories.

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FILTER ADAPTATION

Estimation bias of the type caused by a maneuvering target may be observed building up in the prediction difference term, enclosed in brackets in the filtering equation:

$$\hat{x}_{k|k} = \Phi\hat{x}_{k-1|k-1} + G_k[z_k - \hat{z}_{k|k-1}]. \qquad (1)$$

When the vector z_k is measured with a zero-mean error sequence of known covariance R, the nonmaneuvering, nonrandomly excited prediction difference should have a zero mean and a covariance

$$\begin{aligned}
Pz_{k|k-1} &\triangleq E[(z_k - \hat{z}_{k|k-1})(z_k - \hat{z}_{k|k-1})^T] \\
Pz_{k|k-1} &= H\Phi P_{k-1|k-1}\Phi^T H^T + R
\end{aligned} \qquad (2)$$

where H is the measurement matrix, Φ the transition matrix, and

$$\begin{aligned}
P_{k-1|k-1} = E[(x_{k-1} - \hat{x}_{k-1|k-1}) \\
\cdot (x_{k-1} - \hat{x}_{k-1|k-1})^T]
\end{aligned} \qquad (3)$$

the theoretical estimation error covariance for the $(k-1)$th sample. Reasonable tests for bias can be performed by comparing, at every sample and for strings of consecutive samples, the magnitudes and signs of the elements of the $z_k - \hat{z}_{k|k-1}$ difference vector relative to the appropriate variance terms on the diagonal of the covariance matrix $Pz_{k|k-1}$. For example, should two or more consecutive differences be of the same sign and outside the limits of a 3σ gate, a very strong indication exists that the target is undergoing a maneuver. To recover from the bias introduced by such a maneuver, it is obvious that the raw observation data must be weighted more heavily than would be the case if the subsequent filter gains were taken from the routine gain schedule This suggests a backsliding in the schedule and a reprocessing of the most recent several measurements, the mechanics of which are now described. Assume that the kth measurement has just been taken. Assume also that the bias detector bases its maneuver decisions on the most recent n measurements, including the kth. If upon processing the kth measurement the bias detector concludes that a maneuver is in progress, the filter is to backstep to the $(k-n+1)$th time point and reinitialize its state vector to the value

$$\hat{x}_{k-n+1|k-n} = \Phi\hat{x}_{k-n|k-n}. \qquad (4)$$

The filter is then to reprocess the measurement z_{k-n+1} (presumed to have been stored for this eventuality) using the filtering equation

$$\begin{aligned}
\hat{x}_{k-n+1|k-n+1} = \hat{x}_{k-n+1|k-n} \\
+ G_N[z_{k-n+1} - \hat{z}_{k-n+1|k-n}]
\end{aligned} \qquad (5)$$

where the integer N indicates a certain point in the routine gain schedule. The value of N is generally to be set quite low so that the relatively high gains of the early part of the schedule are brought to bear upon the most recent measurements, those thought to be taken during a target maneuver. The reprocessing continues as outlined above, using G_{N+1}, G_{N+2}, etc., until the n most recent measurements are reprocessed, whereupon normal filtering and maneuver detection processes are resumed. The filter gain, however, is not restored to its premaneuver point in the schedule, but proceeds sequentially from the backstep point as indicated above.